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MINIMAX SYNTHESIS IN PROBLEMS OF PULSE GUIDANCE AND MOTION CORRECTION

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Problems of pulse guidance and motion correction are considered under conditions of indeterminacy on the basis of initial data and perturbations in the system of measurement (observation) of phase coordinates. Guaranteed unimprovable estimates of the minimax miss of the system are obtained in a linear approximation. Estimate of the optimum number of observations and of pulse control effects is indicated. By its statement and method of solution derivation this work is closely related to the investigations in [1 — 3].

1. General statement of the problem. Let the derivation of the controlled object from the specified path $x^\circ(t) \equiv 0$ during the interval of time $t_0 \leqslant t \leqslant \vartheta$ be defined by the equation of linear approximation

$$dx/dt = A(t)x + B(t)u \tag{1.1}$$

where x is an n-dimensional phase vector, $A(t)$ and $B(t)$ are continuous in $[t_0, \vartheta]$ matrices of order $n \times n$ and $n \times r$ respectively, and u an r-dimensional control vector subjected to the restriction

$$u = dU/dt, \quad \int_{t_0}^{\vartheta} \|dU(t)\| \leqslant \mu, \quad \mu - \text{const} > 0 \tag{1.2}$$

Here and below the symbol $\|q\|$ denotes the Euclidean norm of vector q. Let the indicated deviation be assessed by the quantity $r(Nx(t))$ $(r(0) = 0)$, where N is a constant $(k \times n)$-matrix and $r(\cdot)$ is some function of phase coordinates specified on $R^{(k)}$.

The aim of the control is to select action $u(t)$ (1.2) that would ensure the minimum miss $r(Nx(\vartheta))$ of object (1.1) on condition that the information on the initial state is limited to $x(t_0) \in X^\circ$. Region X° is assumed to be convex and closed, and may coincide with the complete phase space.

To define more accurately the phase state of object (1.1), we measure a certain m-dimensional vector y whose relation to the phase vector x is defined by the equation of linear approximation.

$$y = G(t)x + \xi(t) \tag{1.3}$$

where $G(t)$ is a known continuous $(m \times n)$-matrix and $\xi(t)$ is the interference in the observation channel. The models of interference $\xi(t)$ are not a priori specified but are subjected to restriction

$$\varphi(\xi(t)) \leqslant 0, \quad t_0 \leqslant t \leqslant \vartheta \tag{1.4}$$

where $\varphi(\cdot)$ is a specified function (e.g. (1.4)) that can be a restriction of the form $\| \xi(t) \| \leqslant \nu$ imposed on the magnitude of the interference). We assume that system (1.1),(1.3) for $u(t) = \xi(t) \equiv 0$ is entirely observable during any time interval $[t_0, t]$, $t_0 \leqslant t \leqslant \vartheta$.

The processing of signal $y_t^*(\cdot)$ $(y_t^*(\cdot) = y^*(\tau),\ t_0 \leqslant \tau \leqslant t)$ (1.3),(1.4) received during the time interval $t_0 \leqslant \tau \leqslant t$, i.e. the solution of the problem of observation makes it possible to establish a certain set $X(t, y_t^*(\cdot)) \subset R^{(n)}$ which is the region where the phase vector $x(t)$ of system (1.1) remains during each current instant of time t. The programed control $u(t)$ selected for the interval $[t, \vartheta]$ must take into account all possible trajectories simultaneously released from $X(t, y_t^*(\cdot))$. This leads to the problem of control of a set of trajectories. The solutions of the problem of control and observation are in fact separated here: the observation process precedes that of control and the intervals of control and observation do not overlap.

If, however, instant t of completion of observation is not fixed, the incompleteness of information about the initial state and on moving coordinates of the system leads to the problem of simultaneous optimization of the control and observation processes and, among other things, to the synthesis of these at the instant of transition from observation to control.

The aim of this paper is to present an exact description of the solution and to obtain a guaranteed unimprovable estimate of the miss, as well to estimate the number of pulse observations and control. Problems of this kind were considered in [1 — 5], where various methods of approximate solution of the problem were proposed.

2. Basic definitions and assumptions. We call admissible the control defined by functions $U(t)$ of limited variation that are continuous from the right along $[t_0, \vartheta]$ and satisfy restriction (1.2). We assume *a priori* that the models of $\xi(t)$ are piecewise continuous functions which for definiteness are assumed to be continuous from the right. We denote by Ξ_1 the set of all piecewise continuous and continuous from the right m-vector functions in $[t_0, \vartheta]$ that satisfy condition (1.4), and assume that Ξ is a subset of Ξ_1 consisting of continuous functions.

We denote by $X(t, \cdot) = X(t, y_t^*(\cdot) \mid U^*(t))$ of such, and only such, vectors $x = x(t)$ which can obtain at instant t because of the trajectory $x(\tau)$ of system (1.1) for some $x(t_0) \in X^\circ$ and a fixed admissible control $U^*(\cdot)$ $(U(t_0) = 0)$ in $[t_0, t]$ under condition that each of the functions $x(\tau)$ paired with some model $\xi(\tau) \in \Xi_1$, $t_0 \leqslant \tau \leqslant t$, generates signal $y^*(\tau)$ (by formula (1.3)).

We denote by $X^\vartheta(U(\cdot) \mid X(t, \cdot))$ the set of such, and only such, vectors $x = x(\vartheta)$ which can obtain at instant ϑ because of the trajectory $x(\tau)$ of system (1.1) for $x(t) \in X(t, \cdot)$ and fixed control $U(\tau)$ in $[t, \vartheta]$. The control $U(\cdot)$ is assumed here to be admissible. We assume that for $t = \vartheta$ the set $X^\vartheta(U(\cdot) \mid \cdot)$ is equal $X(\vartheta, \cdot) + B(\vartheta)p$, where vector p must satisfy condition

$$\|p\| \leqslant \mu - \int_{t_0}^{\vartheta} \| dU^*(\tau) \|$$

We denote by $Y(t_1, y_t^*(\cdot) \mid U^*(\cdot))$ the set of all possible continuations into the interval $[t_0, t_1]$ of signal $y_t^*(\cdot)$ obtained in $[t_0, t]$ that are admitted by relationships (1.1) — (1.4) for the specified admissible control $U^*(\tau)$, $t_0 \leqslant \tau \leqslant t_1$. Each of such continuations is uniquely determined by specifying vector $x(t) \in X(t, \cdot)$ and function $\xi(\tau) \in \Xi_1, \; t \leqslant \tau \leqslant t_1$.

We make the following assumption.

Assumption 2.1. (a) On $R^{(k)}$ function $r(\cdot)$ is nonnegative, finite, and convex; (b) function $\varphi(\cdot)$ specified on $R^{(m)}$ is convex, finite, and has no recessional directions ([6], p. 86) and $\varphi(0) < 0$; (c) the cut-in $\{x \mid G(t_0)\, x \in y^*(t_0) - Q\} \subseteq X^\circ$, where $Q = \{x \mid \varphi(x) \leqslant 0\}$ is a set of the level of function $\varphi(\cdot)$, takes place at the initial instant t_0.

Note that Assumption 2.1, b implies the boundedness of set Q. Furthermore, according to [6], $0 \in \operatorname{int} Q$ ($\operatorname{int} Q$ is the set of inner points of set Q). The support function of set Q is determined by formula (see [6], p. 136)

$$\gamma(x) = \rho(x \mid Q) = \max_q q'x = \operatorname{cl} \psi(x) \tag{2.1}$$

$$\psi(x) = \inf \{\lambda \varphi^*(\lambda^{-1}x) \mid \lambda > 0\}$$

where $\varphi^*(\cdot)$ is a convex function conjugate of $\varphi(\cdot)$ [6], and $\operatorname{cl} \psi(\cdot)$ denotes the closure of function $\psi(\cdot)$.

3. The problem of correction with fixed instant of observation completion. The instant of completion of observation is taken as fixed.

Problem 3.1. We have to determine the quantity

$$r_1^\circ = \min_{U(\cdot)} \max_x r(Nx) = r^\circ(t_1, \; y_{t_1}^*(\cdot)) \tag{3.1}$$
$$x \in X^\vartheta(U(\cdot) \mid X(t_1, \cdot))$$

and the related admissible optimum control $U^\circ(\tau) = U^\circ(\tau \mid y_{t_1}^*(\cdot)), t_1 < \tau \leqslant \vartheta$ which provides the minimum in (3.1) on condition that the control $U^*(\tau)$ for $\tau \leqslant t_1$ and the instant $t_1, \; t_0 \leqslant t_1 \leqslant \vartheta$ are fixed.

Note that for $t_1 = \vartheta$ the selection of control $U^\circ(\tau)$ in (3.1) reduces in conformity with the definition of set $X^\vartheta(U(\cdot) \mid X(t, \cdot))$ to that of choosing the jump of function $U^*(\cdot)$ at instant $\vartheta + 0$. The following lemma defines the set $X(t, \cdot)$ in terms of its support function.

Lemma 3.1. If Assumption 2.1 is admitted, the support function of the convex compact set $X(t, \cdot)$ caused by the continuous variation of signal $y_t^*(\cdot)$ is defined by formula

$$\max_{x \in X(t,\cdot)} l' \cdot x = \rho(l \mid X(t, \cdot)) = \inf_{L(\cdot)} \Big\{ \int_{t_0}^{t} (\gamma[-dL(\tau)] + \tag{3.2}$$
$$dL(\tau)[y^*(\tau) - G(\tau)\, x(\tau; U^*(\cdot))]) \Big\}$$

$$L(\cdot) \in \Lambda(t, l) = \Big\{ L(\cdot) \,\Big|\, \int_{t_0}^{t} dL'(\tau)\, G(\tau)\, S(t, \tau) = l' \Big\} \tag{3.3}$$

where $x(\tau; U^*(\cdot))$ is the solution of system (1.1) with the boundary condition $x(t) = 0$; $\gamma[\cdot]$ is a function determined by formula (2.1), and $S(t, \tau)$ is the fundamental matrix of the conjugate system $s^\cdot = -sA(t)$. The lower bound in formula (3.2) is taken over all m vector functions $L(\cdot)$ of limited variation belonging to the set $\Lambda(t, l)$ in (3.3).

Note 3.1. Lemma 3.1 remains valid if the lower bound in formula (3.2) is taken over all functions $L(\cdot) \in \Lambda(t, l)$ whose generalized derivative is of the form

$$\frac{dL(\tau)}{d\tau} = \sum_{i=1}^{n+2} \alpha_i \delta(\tau - \tau_i) \tag{3.4}$$

where $\alpha_i \in R^{(m)}$, $\tau_i \in [t_0, t]$, $i = 1, \ldots, n+2$. If observations of signal $y_t^*(\cdot)$ are not carried out continuously but only on a certain set $E \subseteq [t_0, t]$, the support function of set $X(t, \cdot)$ is also determined by formula (3.2), where the lower bound must be taken over all functions $L(\cdot) \in \Lambda(t, l)$ of the form (3.4) for $\tau_i \in E$. Hence the set $X(t, \cdot)$ can be defined by formula

$$X(t, \cdot) = \bigcap_{\tau \in E} \{x \mid G(\tau) S(t, \tau) x \in y^*(\tau) - G(\tau) x(\tau; U^*(\cdot)) - Q\} \tag{3.5}$$

The proof of Lemma 3.1 and of the statement in Note 3.1 may be obtained by using the method described in [7] and the result in [8].

If Assumption 2.1, c is not satisfied, the set is determined by the relationship $X(t, \cdot) = \mathrm{X}(t, \cdot) \bigcap S(t_0, t) X^\circ$, where $\mathrm{X}(t, \cdot)$ is a set that is determined in conformity with (3.2), (3.5).

From Lemma 3.1 and Note 3.1 we obtain the following statement.

Lemma 3.2. If signal $y_t^*(\cdot)$ is specified, then for any number $\varepsilon > 0$ and vector $l \in R^{(n)}$ it is possible to find a collection of points $\{\tau_i\} \subseteq [t_0, t]$ with $i = 1, \ldots, n+2$, where n is the dimension of system (1.1), such that

$$\rho(l \mid X_*(t, \cdot)) < \rho(l \mid X(t, \cdot)) + \varepsilon \tag{3.6}$$

where $X(t, \cdot)$ is the set obtained by continuous observation, and $X_*(t, \cdot)$ is a set of form (3.5) obtained by discrete observations at points $\{\tau_i\}$.

Note that formula (3.2) can also be written as

$$\rho(l \mid X(t, \cdot)) = \inf \{\chi(c, l) + c \mid c \in R^{(1)}\} \tag{3.7}$$

$$\chi(c, l) = \inf_{L(\cdot)} \int_{t_0}^{t} \gamma[-dL(\tau)] \tag{3.8}$$

In (3.8) the lower bound is taken over all functions of limited variation $L(\cdot) \in \Lambda(t, l)$ for which

$$\int_{t_0}^{t} dL(\tau)(y^*(\tau) - G(\tau) x(\tau; U^*(\cdot))) = c$$

Note that when the interference $\xi^*(\tau)$ in signal $y_t^*(\cdot)$ belongs to class Ξ, i.e. is a continuous function, then the lower bound in (3.8) is reached for any c on function $L(\cdot)$ of form (3.4) in which $n+1$ has been substituted for $n+2$. It is also possible to show that when

$$y^*(\tau) - G(\tau)(S(t, \tau) p + x(\tau; U^*(\cdot))) \in \operatorname{int} Q, \quad t_0 \leqslant \tau \leqslant t$$

then for some $\mathring{p} \in R^{(n)}$ the lower bound is reached in formula (3.2) (and also in (3.7)). Hence in shuch cases we have, instead of inequality (3.6), the equality $\rho(l \mid X_*(t, \cdot)) = \rho(l \mid X(t, \cdot))$.

Let us revert to the problem formulated at the beginning of Sect. 3. Transposing $\min_{U(\cdot)}$ and $\max_x$ in formula (3.1) and taking into account the definition of set $X^\vartheta(U(\cdot) \mid X(t, \cdot))$, we obtain formula

$$r_1^\circ = r^\circ(t_1, y_{t_1}^*(\cdot)) = \max_{l \in R^{(k)}} \{-\mu^* \max_{t_1 \leqslant \tau \leqslant \vartheta} \|s(\tau; l) B(\tau)\| + (\operatorname{conc} f)(t_1; l)\} \tag{3.9}$$

$$f(t_1;\ l) = \rho\,(s\,(t_1;\ l)\mid X\,(t_1,\ \cdot)) - r^*(l)$$

$$\mu^* = \mu - \int_{t_0}^{t_1} \|dU(\tau)\| \geqslant 0$$

where $s(\tau;\ l)$, $t_1 \leqslant \tau \leqslant \vartheta$ is the solution of system $\dot{s} = -sA\,(t)$ with the boundary condition $s\,(\vartheta) = l'N$. Function $r^*(l)$ is convex on $R^{(k)}$ and conjugate to $r\,(l)$ [6], and $(\operatorname{conc} f)(l)$ denotes the concave envelope of function $f\,(l)$ on $R^{(k)}$.

The following theorem is valid.

Theorem 3.1. An optimum control $U^\circ(\tau)$ for Problem 3.1 always exists and satisfies the principle of minimum

$$\int_{t_1}^{\vartheta} s\,(\tau;\ l^\circ)\,B\,(\tau)\,dU^\circ(\tau) = \min\left\{\int_{t_1}^{\vartheta} s\,(\tau;\ l^\circ)\,B\,(\tau)\,dU\,(\tau)\ \Big|\ \operatorname{Var} U\Big|_{t_1}^{\vartheta} \leqslant \mu^*\right\} \quad (3.10)$$

where l° is the vector for which maximum is reached in formula (3.9). Moreover the optimum control $U^\circ(\tau)$ can be presented in the form

$$u^\circ(\tau) = \frac{dU^\circ(\tau)}{d\tau} = \sum_{i=1}^{k} \mu_i\delta\,(\tau - \tau_i), \quad \sum_{i=1}^{k} \|\mu_i\| \leqslant \mu^* \quad (3.11)$$

$$\tau_i \in [t_1, \vartheta], \quad \mu_i \in R^{(r)}, \quad i = 1, \ldots, k$$

where k is the number of rows of matrix N.

Note 3.2. If matrix $N = n'$ (row-vector), i.e. $k = 1$ and $r\,(Nx) = |\,n'x\,|$, then

$$(\operatorname{conc} f)\,(t_1;\,l) = \begin{Bmatrix} al, & 1 \geqslant l \geqslant 0 \\ -bl, & 0 > l \geqslant -1 \\ -\infty, & |l| > 1 \end{Bmatrix}, \quad \text{if} \quad a + b \leqslant 0$$

$$(\operatorname{conc} f)\,(t_1;\,l) = \begin{Bmatrix} {}^1/_2\,[(a - b)\,l + a + b], & |l| \leqslant 1 \\ -\infty & |l| > 1 \end{Bmatrix}, \quad \text{if} \quad a + b > 0$$

$$a = \rho\,(n'S\,(t_1,\ \vartheta)\mid X\,(t_1,\ \cdot)), \quad b = \rho\,(-n'S\,(t_1,\ \vartheta)\mid X\,(t_1,\ \cdot))$$

Formula (3.9) can be rewritten as

$$r_1^\circ = \max\left\{\frac{a+b}{2},\, a + c,\, b + c\right\},\ c = -\mu^* \max_{t_1 \leqslant \tau \leqslant \vartheta} \|n'S\,(\tau, \vartheta)\,B\,(\tau)\| \quad (3.12)$$

The optimum control $U^\circ(\tau)$ has in this case only one jump.

Let us choose vector $q \in R^{(k)}$ so that

$$\mu^* \max_{t_1 \leqslant \tau \leqslant \vartheta} \|s\,(\tau;\,l)\,B\,(\tau)\| \geqslant l'q \geqslant (\operatorname{conc} f)\,(t_1;\,l), \qquad \forall l \quad (3.13)$$

where r_1° is determined by formula (3.9). Vector q' that satisfies the inequality (3.13) must necessarily exist. On the other hand a direct computation will show that vector q is the solution of the following extremum problem $g\,(l) = (-f\,(t_1;\ l))^*$:

$$r_1^\circ = g\left(\int_{t_1}^{\vartheta} NS\,(\tau, \vartheta)\,B\,(\tau)\,dU^\circ(\tau)\right) = \min_{U(\cdot)} g\left(\int_{t_1}^{\vartheta} NS\,(\tau, \vartheta)\,B\,(\tau)\,dU\,(\tau)\right) \quad (3.14)$$

$$q = -\int_{t_1}^{\vartheta} NS\,(\tau, \vartheta)\,B\,(\tau)\,dU^\circ(\tau) \quad (3.15)$$

We have the following theorem.

Theorem 3.2. If vector $q \in R^{(k)}$ satisfies inequality (3.13), then the control $U^\circ(\tau)$ that resolves problem (3.15) is optimum for Problem 3.1. This control also solves problem (3.14), and any solution of the latter is a solution of problem (3.1).

Here (3.14) can be considered as a problem of minimization of function $g(Nx(\vartheta))$ (3.15) at the final state $x(\vartheta)$ of system (1.1) for a known initial state $x(t_1) = 0$. Theorem 3.2 states that problems 3.1 and (3.14) are equivalent.

Note that the quantity r_1° defined by formula (3.9),(3.14) is the guaranteed result of control for continuous observation of signal (1.3) in $[t_0, t_1]$ and for an exact estimate of region $X(t_1, \cdot)$. If observations are carried out at discrete instants of time or when region $X(t_1, \cdot)$ is only approximately known, the guaranteed result r_1° is less satisfactory.

Let us consider in greater detail one method of estimating region $X(t_1, \cdot)$. Actually, we have to estimate region $NS(t_1, \vartheta)X(t_1, \cdot)$ in $R^{(k)}$, since it is this region that appears in formula (3.9). If perturbations $\xi(\cdot)$ in signal (1.3) are fairly small, region $X(t_1, \cdot)$ lies in some reasonably small neighborhood of the affine set $x^*(t_1) +$ Ker $G(t_1)$ (Ker $G = \{x \mid Gx = 0\}$), where $x^*(t_1)$ is the value of the phase vector of system (1.1) that obtains at instant t_1. With this in mind it is expedient to estimate region $NS(t_1, \vartheta)X(t_1, \cdot)$ by a k-dimensional rectangle Π oriented with respect to the orthogonal axes $l_1, \ldots, l_k$ so that the support functions of sets Π and $NS(t_1, \vartheta)X(t_1, \cdot)$ coincide on unit vectors $\pm l_i$, $i = 1, \ldots, k$, or differ only slightly from these. Vectors $l_1, \ldots, l_k$ are to be chosen so that the first j vectors form the basis in the subspace $NS(t_1, \vartheta)$ Ker $G(t_1)$ and the remaining $k - j$ vectors supplement the former to the orthonormal basis in $R^{(k)}$. This construction shows that the direction of vectors $l_1, \ldots, l_k$ chosen in this manner depends only on the instant of time t_1, while being independent of the availability of signal $y_{t_1}(\cdot)$ and of the configuration of region $X(t_1, \cdot)$.

We introduce the notation

$$f(t_1; l_i) \equiv \rho(s(t_1; l_i) \mid X(t_1, \cdot)) = c_i, \quad f(t_1, -l_i) = c_{k+i} \quad i = 1, \ldots, k$$

and assume that the unit vectors l_i have been chosen by the method indicated above.

It is not difficult to verify that vectors l_i and the quantities c_i uniquely define the k-dimensional rectangle

$$\Pi = L\Pi_* + Lb, \quad L = [l_1, \ldots, l_k], \quad b = \left(\frac{c_1 - c_{k+1}}{2}, \ldots, \frac{c_k - c_{2k}}{2}\right)$$

$$\Pi_* = \left\{\alpha \in R^{(k)} \mid \alpha = (\alpha_1, \ldots, \alpha_k), |\alpha_i| \leqslant \nu_i = \frac{c_i + c_{k+i}}{2}, i = 1, \ldots, k\right\}$$

where L is a matrix composed of column-vectors l_i and Π_* is the k-dimensional rectangle.

Taking the above into consideration, we substitute for (3.9) the approximate formula

$$r_1^\circ = r^\circ(t_1, y_{t_1}^*(\cdot)) = \max_{l'l \leqslant 1} \{l'b - \mu^* \max_{t_1 \leqslant \tau \leqslant \vartheta} \| s(\tau; L'l) B(\tau) \| + (\operatorname{conc} \rho)(l \mid \Pi_*)$$

$$r(l) = \|l\|, \quad r^*(l) = \left\{\begin{array}{ll} 0, & \|l\| \leqslant 1 \\ +\infty, & \|l\| > 1 \end{array}\right\} \qquad (3.16)$$

Function $(\operatorname{conc} \rho)(l \mid \Pi_*)$ consists of $k(k-1) + 1$ pieces of smooth surfaces, and in the k-dimensional rectangle

$$|l_i| \leqslant \nu_i S^{-1}, \quad S = \Big(\sum_{i=1}^{k} \nu_i^2\Big)^{1/2}, \quad i = 1, \ldots, k$$

is constant and equal S.

Below we present the statement on the number of observations required for solving problem (3.16).

Theorem 3.3. For any number $\varepsilon > 0$ it is possible to indicate not more than $2k(n+2)$ points $\{\tau_i\} \subseteq [t_0, t_1]$ such that for the set $X_*(t_1, \cdot)$ obtained by observing signal (1.3) at these points, the result $\bar{r}_1^\circ$ of the solution of the problem (3.16) will satisfy the inequality $\bar{r}_1^\circ < r_1^\circ + \varepsilon$, where r_1° is obtained in the solution of problem (3.16) for the set $X(t_1, \cdot)$ by continuous observation of signal (1.3) in $[t_0, t_1]$.

It is not difficult to obtain the proof of Theorem 3.3 by using Lemma 3.2 and taking into account the continuous dependence of the quantity $(\mathrm{conc}\,\rho)\ (l \mid \Pi)$ on parameters c_i, $i = 1, \ldots, 2k$, which in turn implies the continuous dependence of r_1° (3.16) on these parameters.

4. The problem of correction with synthesis at the instant of completion of observation. Let us consider the following problem.

Problem 4.1. Determine the quantity $r_2^\circ = r^\circ(t^\circ, y_{t^\circ}^*(\cdot))$ and the related admissible optimum control $U^\infty(\tau) = U^\infty(\tau \mid y_{t^\circ}(\cdot))$, $t^\circ < \tau \leqslant \vartheta$, which provides the minimum in (3.1) on condition that the previously specified admissible control $U^*(\tau)$ $(U^*(t_0) = 0)$, $t_0 \leqslant \tau \leqslant \vartheta$, obtains in $[t_0, t^\circ]$. Here $t^\circ = t^\circ(y_{t^\circ}^*(\cdot))$ is the earliest instant of time for which

$$r^\circ(t^\circ, y_{t^\circ}^*(\cdot)) \leqslant \min_{t^\circ \leqslant t \leqslant \vartheta} \sup_{y_t(\cdot)} r^\circ(t, y_t(\cdot)),\ y_t \cdot \in Y(t, y_{t^\circ}^*(\cdot) \mid U^*(\cdot)) \qquad (4.1)$$

In Problem 4.1 the instant of correction is not specified *a priori*. On the contrary, it has to be synthesized according to condition (4.1) on the basis of incoming information. After $t^\circ = t^\circ(y_{t^\circ}^*(\cdot))$ has been determined, Problem 4.1 reduces to Problem 3.1 for $t_1 = t^\circ$. We note in connection with this that in synthesizing that instant of time in Problem 4.1 it is necessary to compute the quantity $r^\circ(t, y_t(\cdot))$ (3.1) for various values of t. The quantity $r^\circ(t, y_t(\cdot))$ may be determined either exactly by formula (3.9) or approximately by formula (3.16) (for $r(l) = \| l \|$). In either case r_2° (4.1) is the guaranteed result of control, and in the worst case for $t_1 \geqslant t^\circ$ we always have $r_1^\circ \geqslant r_2^\circ$. If, however, $t_1 < t^\circ$, the strict inequality $r_1^\circ > r_2^\circ$ is satisfied.

Let us now consider the question of existence of instant $t^\circ(y_{t^\circ}(\cdot))$. For this we need to take into account one property of the linear completely observable systems of the form (1.1), (1.3) $(U(\cdot) = 0)$.

Property A. For any instant $t > t_0$ and any signal (1.3) that is obtained in the interval $[t_0, t]$, as well as for any continuous extension $y_\vartheta(\tau) \in Y(\vartheta, \quad y_t^*(\cdot) \mid 0)$ of signal $y_t^*(\cdot)$, the set $X(\tau, \cdot)$ is in the meaning of Hausdorff's principle continuous at point t from the right.

Without going into the details of this property, we note that for $G(\tau) \equiv G$ (const) system (1.1) possesses property A.

Lemma 4.1. Let Property A be satisfied. Then there exists the smallest instant of time t° for which condition (4.1) is satisfied.

Proof. Let us fix some obtained $y^*(\tau)$, $t_0 \leqslant \tau \leqslant \vartheta$ of signal (1.3), and set

$t^* = \inf\{t^\circ\}$, where $\{t^\circ\}$ is the set of instants t° which for $y_{t\bullet}{}^* (\tau) \equiv y^* (t+\tau)$, $t_0 - t \leqslant \tau \leqslant 0$ satisfy inequality (4.1). Let us now show that at instant t^* inequality (4.1) is satisfied. In fact, specifying number $\varepsilon > 0$ and assuming that $t > t^*$, we can select for each $t^\circ \in \{t^\circ\}$, $t^* < t^\circ < t$ such $y_t(\cdot) \in Y(t, y_{t^\circ}{}^*(\cdot) \mid U^*(\cdot))$ that

$$-\varepsilon + r^\circ(t^\circ, y_{t\bullet}{}^*(\cdot)) < r^\circ(t, y_t(\cdot)) \tag{4.2}$$

in conformity with inequality (4.1). Taking into consideration the continuity of interference $\xi^*(\cdot)$ and of control $U^*(\cdot)$, and allowing for the Property A, we find from formulas (3.9) and (3.16) that function $r^\circ(t, y_t{}^*(\cdot))$ is semicontinuous from below at point t^* when t tends to t^* from the right. Passing in inequality (4.2) to the limit for $t^\circ \to t^* + 0$, we obtain $-\varepsilon + r^\circ(t^*, y_{t*}{}^*(\cdot)) \leqslant r^\circ(t, y_t(\cdot))$. Owing to the arbitrariness of the selection of the number ε and of the instant $t > t^*$, we can consider that the validity of inequality (4.1) at instant t^* is established.

The reasoning of Sect. 4 is summarized by the following theorem.

Theorem 4.1. The solution of Problem 4.1, i.e. the pair $\{t^\circ(y_{t^\circ}{}^*(\cdot))$ and $U^\infty(\cdot \mid y_{t^\circ}{}^*(\cdot))\}$ satisfies condition

$$\sup_{y_\vartheta(\cdot)} \max_{x(\vartheta)} r(Nx(\vartheta)) = \min_{\{t, U(\cdot)\}} \sup_{y_\vartheta(\cdot)} \max_{x(\vartheta)} r(Nx(\vartheta)) \tag{4.3}$$

$$y_\vartheta(\cdot) \in Y(\vartheta, \ y_t{}^*(\cdot) \mid U^*(\cdot)), \quad x(\vartheta) \in X^\vartheta(U(\cdot) \mid X(t, \ y_t{}^* \times (\cdot) \mid U^*(\cdot)))$$

The maximum in the left-hand part of equality (4.3) is taken for $t = t^\circ$ and $U(\cdot) = U^\infty$.

5. Example. Let us consider system $(u = dU / dt)$

$$\dot{x}_1 = x_2, \ \dot{x}_2 = u, \ 0 \leqslant t \leqslant \vartheta, \quad \int_0^\vartheta |dU(\tau)| \leqslant \mu \tag{5.1}$$

We observe the signal

$$y(\tau) = x_1(\tau) + \xi(\tau), \ \ |\xi(\tau)| \leqslant \Delta, \ \Delta = \text{const} \tag{5.2}$$

The deviation of motion $x(t)$ from the specified $x^\circ(t) \equiv 0$ is assessed by the quantity $r(Nx(t)) = |x_1(t)|$. We impose in the general case the additional restriction on the motion of system (5.1) by specifying that at the final instant of time the coordinates must satisfy the condition

$$|x_2(\vartheta)| \leqslant \nu, \quad \nu = \text{const} \tag{5.3}$$

Problem 3.1 for system (5.1) with allowance for restriction (5.3) is formulated as follows: determine the minimum number α° and the optimum control $U^\circ(\tau)$, $t < \tau$, such that the inequalities $|x_1(\vartheta)| < \alpha^\circ$ and $|x_2(\vartheta)| \leqslant \nu$ are satisfied by all vectors $x(\vartheta) \in X^\vartheta(U^\circ(\cdot) \mid X(t, \cdot))$.

The numerical solution of the problem is carried out for $\mu = 6$, $\nu = 8$, $\Delta = 2$, $\vartheta = 2$, $X^\circ = \{(x_1, x_2) \mid -14 \leqslant x_1 \leqslant -6\}$ $U^*(\tau) \equiv 0$, and the simulated signal $y(\tau)$ and interference $\xi(\tau)$ are specified in the form (Fig. 1)

$$y^*(\tau)=\begin{cases}-8+4\tau, & 0\leqslant\tau<0.5\\ -9+4\tau, & 0.5\leqslant\tau<1.0\\ -12+4\tau, & 1.0\leqslant\tau<1.5\\ -8+4\tau, & 1,5\leqslant\tau\leqslant2.0\end{cases},\qquad \xi^*(\tau)=\begin{cases}2, & 0\leqslant\tau<0.5\\ 1, & 0.5\leqslant\tau<1.0\\ -2, & 1.0\leqslant\tau<1.5\\ 2, & 1.5\leqslant\tau\leqslant2.0\end{cases}\qquad (5.4)$$

The regions $X(t,\cdot)$ which correspond to the continuous observation of signal (5.4) (formula (3.5) is the most convenient for computing these) are represented by polygons. Regions $X(t,\cdot)$ in Fig. 2 denoted by numerals 1-6 relate to instants of time $0.5-0$, 0.5, $1-0$, 1.0, $1.5-0$, 1.5, respectively. The first two sets are represented by parallelograms and the last three by irregular polygons. At instant $t=1.5$ region $X(t,\cdot)$ contracts abruptly to the point at coordinates $(-4,4)$. Note that for computing the set $X(0.5-0,\cdot)$ it is sufficient to make only two observations of signal (5.4) (namely at $t=0$ and $t=0.5-0$), for set $X(0.5,\cdot)$ three measurements (at $t=0$, $t=0.5-0$ and $t=0.5$) are necessary, and for the sets $X(1-0,\cdot)$ and $X(1,\cdot)$ four and five measurements (at $t=0$, $t=0,5-0$, $t=0,5$ and so on), respectively, are required.

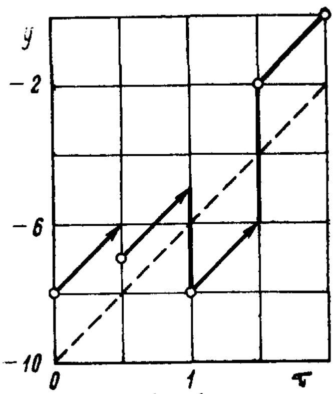

Fig. 1

Let us solve problem (3.1), (5.1) for instant $t_1=0,5-0$. Ignoring restrictions (5.3) and using formula (3.12), we obtain $n^0=\alpha_{\min}=14$. We further note that the control $u^0(\tau)=-4\delta(\tau-2)$ solves the problem for $\alpha^0=14$ $(\nu=8)$. Thus the minimum with respect to x_1 neighborhood of zero is $\alpha^0=14$ for $t_1=0.5-0$. Similar computations for $t_1=0.5$

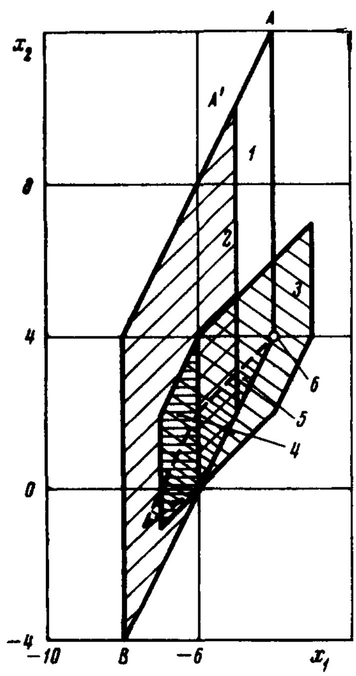

Fig. 2

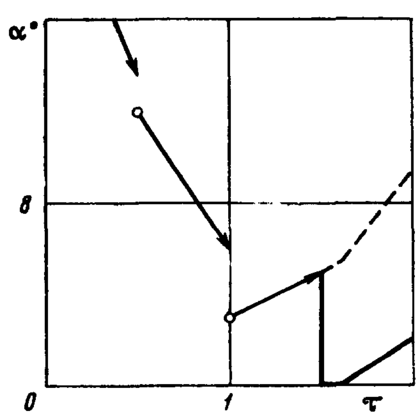

Fig. 3

yield $\alpha^0=12$ for the control $u^0(\tau)=2\delta(\tau-1)-4\delta(\tau-2)$. It should be noted that the problem of transfering sets $X(0.5-0,\cdot)$ and $X(0.5,\cdot)$ to the zero neighborhood that is minimum with respect to x_1 is equivalent to the problem of transfer to the same neighborhood of the segments which connect vertices $A - B$ and $A' - B$ of polygons $X(0.5-0,\cdot)$ and $X(0.5,\cdot)$ (see Fig. 2).

Let us consider now Problem 4.1. First, we point out that one must not complete the observation earlier than at $t = 1$. Here, even in the worst case of obtained signal $y(\tau)$, $\tau \geqslant t$, a further observation yields a lower value for α°. Thus, solving Problem 3.1 ((5.1.)) for $t_1 = 1$ and control $u^\circ(\tau) = 5\delta(\tau - 1) - \delta(\tau - 2)$ we find that $\alpha^\circ = 3$. In the case of the "worst" signal $y(\tau) = -12 + 4\tau$, $\tau > 1$ region $X(t,\cdot)$ that corresponds to that signal is obtained by a simple transformation of set $X(1,\cdot)$ on the basis of system (5.1). Formula (3.12) implies that for all t, $1 \leqslant t \leqslant 2$ we have $\alpha^\circ \geqslant r_1 \geqslant 3$. Thus $t^\circ = t(y^*(\cdot)) = 1$ and the control $u^\infty(\tau) = 5\delta(\tau - 1) - \delta(\tau - 2)$ provide the solution of problem (4.1) for system (5.1),(5.2) for a specific signal (5.4). The minimum value of α° that can be guaranteed on the basis of incoming information (5.4) is equal three. The continuation of observation beyond $t = 1$ would in the worst case of $t = 1.5$ yield $\alpha^\circ = 5$. In the case of signal (5.4) obtained at instant $t = 1.5$ region $X(t,\cdot)$ would contract to a point, and the control $u^\circ(\tau) = 4\delta(\tau - 1.5)$ would yield $\alpha^\circ = 0$ (Fig. 3).

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